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New localized Superluminal solutions to the wave equations with finite total energies and arbitrary frequencies*

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Abstract. By a generalized bidirectional decomposition method, we obtain new Superluminal localized solutions to the wave equation (for the electromagnetic case, in particular) which are suitable for arbitrary frequency bands; several of them being endowed with *finite* total energy. We construct, among the others, an infinite family of generalizations of the so-called "X-shaped" waves. Results of this kind may find application in the other fields in which an essential role is played by a wave-equation (like acoustics, seismology, geophysics, gravitation, elementary particle physics, etc.).

PACS. 03.50.De Classical electromagnetism, Maxwell equations – 41.20.Jb Electromagnetic wave propagation; radiowave propagation

1 Introduction

Since many years it has been known that localized (non-dispersive) solutions exist to the wave equation [1], endowed with subluminal or Superluminal [2] velocities.

Particular attention has been paid to the localized Superluminal solutions, which seem to exist and propagate not only in vacuum but also in media with boundaries [3], like normal-sized metallic waveguides [4] and possibly optical fibers.

It is well-known that such Superluminal localized solutions (SLS) have been *experimentally* produced in acoustics [5], in optics [6] and recently in microwave physics [7].

However, all the analytical SLSs considered till now and known to us, with one exception [8], are superposition of Bessel beams with a frequency spectrum starting from $\nu = 0$ and suitable for low frequency regions. In this paper we shall set forth a new class of SLSs with a spectrum beginning at any arbitrary frequency, and therefore well suited for the construction also of high frequency (microwave, optical, ...) pulses.

2 "V-cone" variables: a generalized bidirectional expansion

Let us start from the axially symmetric solution (Bessel beam) to the wave equation in cylindrical co-ordinates:

$$\psi(\rho, z, t) = J_0(k\rho)\, \mathrm{e}^{+\mathrm{i}k_z z}\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{1}$$

with the conditions

$$k^2 = \frac{\omega^2}{c^2} - k_z^2; \qquad k^2 \geq 0, \tag{2}$$

where J_0 is the zeroth-order ordinary Bessel function, and where (as usual) k_z is the longitudinal component of the wavenumber while $k \equiv k_\perp$ is the wavenumber transverse component magnitude. The second condition (2) excludes the non-physical solutions.

It is essential to stress right now that the dispersion relation (2), with positive (but not constant, *a priori*) k^2 and real k_z, while enforcing the consideration of the truly propagating waves only (with exclusion of the evanescent ones), does allow for both subluminal and Superluminal solutions!; the latter being the ones of interest here for us. Conditions (2) correspond in the (ω, k_z) plane to confining ourselves to the sector shown in Figure 1; that is, when one chooses $\omega \geq 0$ to the region delimited by the straight lines $\omega = \pm ck_z$.

A general, axially symmetric superposition of Bessel beams (with Φ' as spectral weight-function) will

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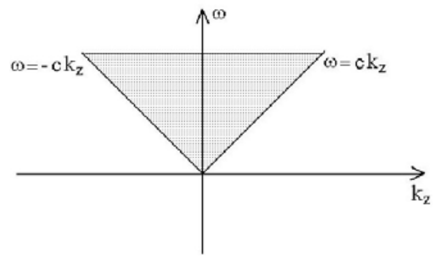


Fig. 1. Geometrical representation, in the plane (ω, k_z), of our conditions (2) for $\omega \geq 0$: see the text. It is essential to notice that the dispersion relation (2), with positive (but not constant, *a priori*) k^2 and real k_z, while enforcing the consideration of the truly propagating waves only (with exclusion of the evanescent ones), does allow for both subluminal and Superluminal solutions; the latter being the ones of interest for us. Conditions (2) correspond to confining ourselves to the sector delimited by the straight lines $\omega = \pm c k_z$.

therefore be:

$$\Psi(\rho, z, t) = \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}\omega \int_{-\omega/c}^{+\omega/c} \mathrm{d}k_z\, \psi(\rho, z, t) \times \delta\left(k - \sqrt{\frac{\omega^2}{c^2} - k_z^2}\right) \Phi'(\omega, k_z; k). \quad (3)$$

Notice that it is $k \geq 0$; $\omega \geq 0$ and $-\omega/c \leq k_z \leq +\omega/c$. The question of the negative k_z values entering expansion (3) will soon be considered below.

The base functions $\psi(\rho, z, t)$ can be however rewritten as

$$\psi(\rho, \zeta, \eta) = J_0(k\rho)\, \exp \mathrm{i}[\alpha\zeta - \beta\eta],$$

where (α, β), which will substitute in the following for the parameters (ω, k_z), are

$$\alpha \equiv \frac{1}{2V}(\omega + V k_z); \qquad \beta \equiv \frac{1}{2V}(\omega - V k_z), \quad (4)$$

in terms of the new "V-cone" variables:

$$\begin{cases} \zeta \equiv z - Vt \\ \eta \equiv z + Vt. \end{cases} \quad (5)$$

The present procedure is a generalization of the so-called "bidirectional decomposition" technique [9], which was devised in the past for $V = c$.

The "V-cone" of Figure 2a corresponds in the (ω, k_z) plane to the straight-lines $\omega \pm V k_z = 0$, that is, to the lines $\alpha = 0$ and $\beta = 0$ (Fig. 2b); while conditions (2) become [let us put $c = 1$ *whenever convenient,* throughout this paper]:

$$k^2 = V^2(\alpha + \beta)^2 - (\alpha - \beta)^2 \equiv (\alpha^2 + \beta^2)(V^2 - 1) + 2(V^2 + 1)\alpha\beta; \quad k^2 \geq 0. \quad (2')$$

Inside the allowed region shown in Figure 1, we can choose for simplicity the sector delimited by the straight-lines $\omega = \pm V k_z$ shown in Figure 2b, *provided that* $V > 1$.

Let us observe that integrating over the intervals $\alpha, \beta \geq 0$ corresponds in equation (3) to integrating over k_z between $-\omega/V$ and $+\omega/V$. But we shall choose in equation (3) spectral weights $\Phi'(\omega, k_z; k)$, and therefore spectral weights $\Phi(\alpha, \beta; k)$ in equation (3') below, such as to either eliminate or make negligible the contribution from the negative values of k_z, that is, from the backwards moving waves: thus curing from the start the problem met by the "bidirectional decomposition" technique in connection with the so-called non-causal components. Therefore, our SLSs will all be physical solutions.

Let us recall also that each Bessel beam is associated with an ("axicone") angle θ, linked to its speed by the relations [10]:

$$\tan\theta = \sqrt{V^2 - 1}; \quad \sin\theta = \frac{\sqrt{V^2 - 1}}{V}; \quad \cos\theta = \frac{1}{V}, \quad (6)$$

where $V \to 1$ when $\theta \to 0$, while $V \to \infty$ when $\theta \to \pi/2$.

Therefore, instead of equation (3) we shall consider the (more easily integrable) Bessel beam superposition in the new variables [with $V > 1$]

$$\Psi(\rho, \zeta, \eta) = \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}\alpha \int_0^\infty \mathrm{d}\beta\, J_0(k\rho)\, \mathrm{e}^{\mathrm{i}\alpha\zeta}\, \mathrm{e}^{-\mathrm{i}\beta\eta} \times \delta\left(k - \sqrt{(\alpha^2 + \beta^2)(V^2 - 1) + 2(V^2 + 1)\alpha\beta}\right) \times \Phi(\alpha, \beta; k) \quad (3')$$

where the integrations over α, β between 0 and ∞ just correspond to the dashed region of Figure 2b. Between the spectral weights Φ of equation (3') and the previous Φ' of equation (3) it holds the relation

$$\Phi(\alpha, \beta; k) = 2V\, \Phi'\left(V(\alpha + \beta),\, \alpha - \beta;\, k\right),$$

quantity $2V$ being just a dispensable multiplicative factor.

For clarity's sake, let us comment a little more on our choice of the integration limits which enter equation (3'). From equation (3) one has that $0 \leq \omega < \infty$ and $-\omega/c \leq k_z \leq \omega/c$. From such inequalities, and from transformations (4) written in the form $k_z = \alpha - \beta$; $\omega = V(\alpha + \beta)$, one easily gets that the integration limits for α and β have to obey the three inequalities

$$0 < \alpha + \beta < \infty \quad (4a)$$

$$(1 + V/c)\, \alpha \geq (1 - V/c)\, \beta \quad (4b)$$

and

$$(1 - V/c)\, \alpha \leq (1 + V/c)\, \beta. \quad (4c)$$

At this point one has to carefully distinguish the case $V < c$ from $V > c$. In our case ($V > c$), we can write equations (4b, 4c) as

$$\alpha \geq \frac{1 - V/c}{1 + V/c}\, \beta \quad (4b')$$

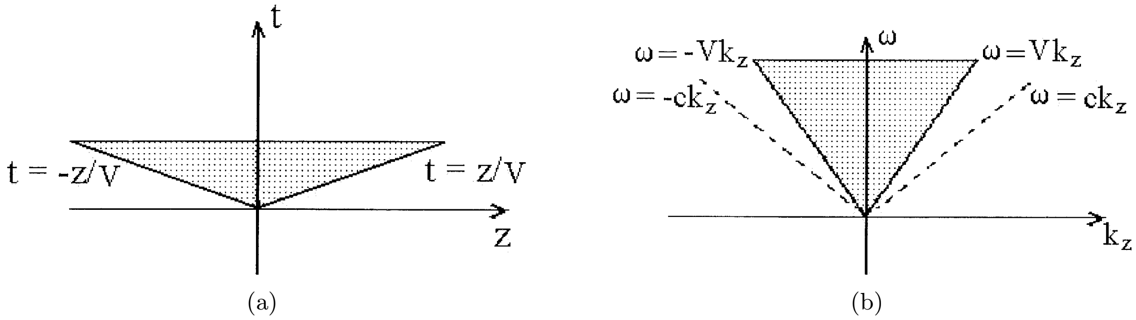


Fig. 2. The "V-cone" (shown in (a)) corresponds in the (ω, k_z) plane to the straight-lines $\omega \pm V k_z = 0$. Inside the allowed region, shown in Figure 1, we choose for simplicity (see the text) the sector depicted in (b). We assume $V > 1$ and confine ourselves to $\omega \geq 0$.

and

$$\alpha \geq \frac{1 + V/c}{1 - V/c}\,\beta. \tag{4c'}$$

Let us finally suppose both α and β to be positive $[\alpha, \beta > 0]$. Inequality (4a) is then satisfied; while the coefficients $(1-V/c)/(1+V/c)$ and $(1+V/c)/(1-V/c)$ entering relations (4b', 4c') are both negatives (since $V > c$). As a consequence, the inequalities (4b', 4c') result to be *automatically* satisfied: this means that we can actually choose $\alpha > 0$ and $\beta > 0$, as we did in equation (3'). In other words, the integration limits of our equation (3') are *contained* by those discussed in connection with equation (3), and are therefore acceptable. Indeed, they constitute a rather suitable choice for facilitating all the subsequent integrations.

We shall now go on to constructing new Superluminal localized solutions for arbitrary frequencies, several of them possessing finite total energy.

3 Some new Superluminal localized solutions for arbitrary frequencies and/or with finite total energy

3.1 The classical "X-shaped solution" and its generalizations

Let us start by choosing the spectrum [with $a > 0$]:

$$\Phi(\alpha, \beta) = \delta(\beta - \beta')\, \mathrm{e}^{-a\alpha}, \tag{7}$$

$a > 0$ and $\beta' \geq 0$ being constants (related to the transverse and longitudinal localization of the pulse).

In the simple case when $\beta' = 0$, one completely dispenses with the "non-causal" (backwards-moving) components of the bidirectional Fourier-type expansion (3'). For the sake of clarity, let us go back to examining Figure 2b: the $\delta(\beta)$ factor in spectrum (7) does actually imply the integrations over α and β in equation (3') to run along the α-line only; *i.e.*, along the $\beta = 0$ straight-line (where $\omega = +Vk_z$). In this case, even more than in the others, it is easy to verify that the group-velocity[1] of the present solution [*cf.* Eq. (8) below] is $\partial\omega/\partial k_z = 1/\cos\theta \equiv V > 1$. Let us, then, choose $\beta' = 0$, and observe that for $\beta = 0$ all the solutions $\Psi(\rho, \zeta, \eta)$ are actually functions only of ρ and $\zeta = z - Vt$. [Let us also notice that in empty space such solutions $\Psi(\rho,\, \zeta = z - Vt)$ can be transversely localized only if $V \neq c$, because if $V = c$ the function Ψ has to obey the Laplace equation on the transverse planes. Let us recall that in this paper we always assume $V > 0$.]

In the present case, equation (3') can be easily integrated over β and k by having recourse to identity (6.611.1) of reference [11], yielding

$$\Psi_{\mathrm{X}}(\rho, \zeta) = \int_0^\infty \mathrm{d}\alpha\, J_0(\rho\alpha\sqrt{V^2 - 1})\, \mathrm{e}^{-\alpha(a - \mathrm{i}\zeta)} = \left[(a - \mathrm{i}\zeta)^2 + \rho^2(V^2 - 1)\right]^{-1/2}, \tag{8}$$

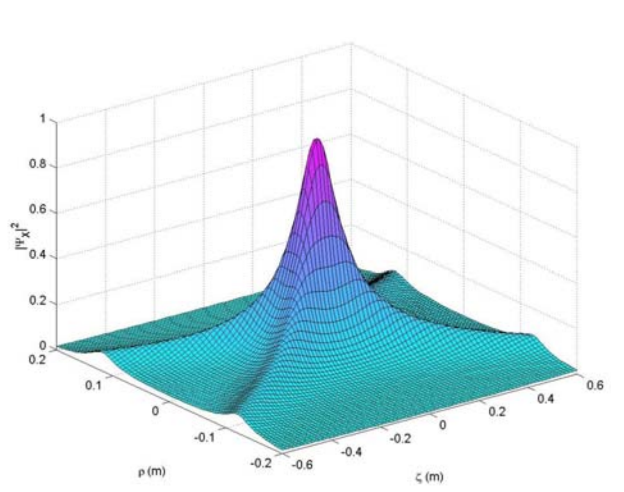
which is exactly the classical X-shaped solution proposed by Lu & Greenleaf [12] in acoustics, and later on by others [12] in electromagnetism, once relations (6) are taken into account. See Figure 3a.

Many other SLSs can be easily constructed; for instance, by inserting into the weight function (7) the extra factor α^m, namely $\Phi(\alpha, \beta) = \alpha^m\, \delta(\beta) \exp[-a\alpha]$, where m is a non-negative integer, while it is still $\beta' = 0$. Then an infinite family of new SLSs is obtained (for $m \geq 0$), by using, this time, identity (6.621.4) of the same reference [11]:

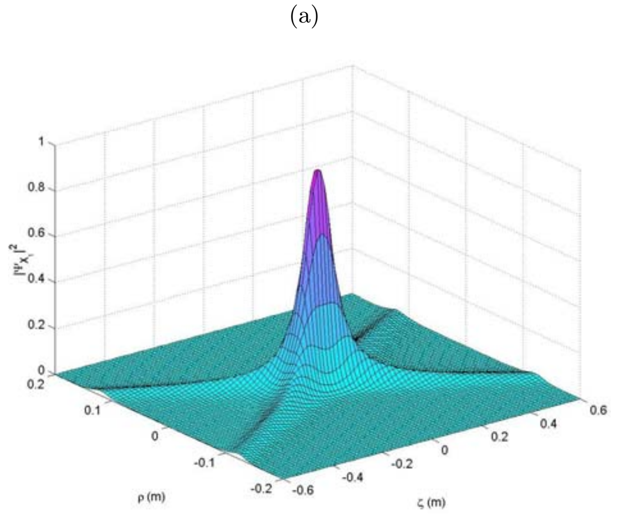
$$\Psi_{\mathrm{X},m}(\rho, \zeta) = (-\mathrm{i})^m \frac{\mathrm{d}^m}{\mathrm{d}\zeta^m} \left[(a - \mathrm{i}\zeta)^2 + \rho^2(V^2 - 1)\right]^{-1/2} \tag{9}$$

which generalize [13] the classical X-shaped solution, corresponding to $m = 0$: namely, $\Psi_{\mathrm{X}} \equiv \Psi_{\mathrm{X},0}$. Notice that all

[1] Let us observe that the group velocity of the solutions considered in this paper can *a priori* be evaluated through the ordinary, simple derivation of ω with respect to the wavenumber only for the infinite total energy solutions, as in the present case. However, for our SSP and SMPS solutions, below, and in general for the finite total energy Superluminal solutions, the group-velocity cannot be calculated through that simple relation, since in those cases it does not even exist a one-to-one function $\omega = \omega(k_z)$.



(a)



(b)

Fig. 3. In (a) it is represented (in arbitrary units) the square magnitude of the "classical", X-shaped Superluminal localized solution (SLS) to the wave equation [12], with $V = 5c$ and $a = 0.1$ m: *cf.* equations (8, 6). An infinite family of SLSs however exists, which generalize the classical X-shaped solution; the figure (b) depicts the first of them (its first derivative) with the same parameters: see the text and equation (10). The successive solutions in such a family are more and more localized around their vertex. Quantity ρ is the distance in meters from the propagation axis z, while quantity ζ is the "V-cone" variable (still in meters) $\zeta \equiv z - Vt$, with $V > c$. Since all these solutions depend on z (and t) only through the variable ζ, they propagate "rigidly", *i.e.*, without distortion (and are called "localized", or non-dispersive, for such a reason). In this paper we assume propagation in the vacuum (or in a homogeneous medium).

the derivatives of the latter with respect to ζ lead to new SLSs, all of them being X-shaped.

In the particular case $m = 1$, one gets the SLS

$$\Psi_{\mathrm{X},1}(\rho,\zeta) = \frac{-\mathrm{i}\,(a - \mathrm{i}\zeta)}{\left[(a-\mathrm{i}\zeta)^2 + \rho^2(V^2-1)\right]^{3/2}} \qquad (10)$$

which is the first derivative of the X-shaped wave, and is depicted in Figure 3b. One can notice that, by increasing m, the pulse become more and more localized around its vertex. All such pulses travel, however, without deforming.

Solution (8) is suited for low frequencies only, since its frequency spectrum (exponentially decreasing) starts from zero. One can see this for instance by writing equation (7) in the (ω, k_z) plane: by equations (4) one obtains

$$\Phi(\omega, k_z) = \delta\left(\frac{\omega - Vk_z}{2V} - \beta'\right)\exp\left[-a\,\frac{\omega + Vk_z}{2V}\right]$$

and can observe that $\beta' = 0$ in the delta implies $\omega = Vk_z$. So that the spectrum becomes $\Phi = \exp[-a\omega/V]$, which starts from zero and has a width given by $\Delta\omega = V/a$.

By contrast, when the factor α^m is present, the frequency spectrum of the solutions can be "bumped" in correspondence with any value ω_{M} of the angular frequency, provided that m is large [or a/V is small]: in fact, ω_{M} results to be $\omega_{\mathrm{M}} = mV/a$. The spectrum, then, is shifted towards higher frequencies (and decays only beyond the value ω_{M}).

Moreover, let us mention here that also in the spectra of the following pulses (considered in Sects. 3.2 and 3.3) one can insert the α^m factor; in fact, in correspondence with the spectrum

$$\Phi(\alpha,\beta) = \alpha^m\,\Phi_0(\beta)\,\mathrm{e}^{-a\alpha}, \qquad (7')$$

one obtains *as further solutions* the mth order derivatives of the basic ($m = 0$) solution below considered. This is due to the circumstance that our integrations over α (as in Eq. (3')) are always Laplace-type transformations. We shall not write them down explicitly, however, for the sake of conciseness.

Different SLSs can be obtained also by *modifying* (still with $\beta' = 0$) the spectrum (7). Some interesting solutions are reported in Appendix A.

Let us now construct SLSs more suited for high frequencies (always confining ourselves to pulses well localized not only longitudinally, but also transversely).

3.2 The Superluminal "focus-wave modes" (SFWM)

Let us go back once more to spectrum (7), but examining now the general case with $\beta' \neq 0$. After integrating over k and β, equation (3') yields [$a > 0$; $\beta' > 0$; $V > c$]:

$$\Psi(\rho,\zeta,\eta) = \mathrm{e}^{-\mathrm{i}\beta'\eta}\int_0^\infty \mathrm{d}\alpha \times J_0\left(\rho\sqrt{V^2(\alpha+\beta')^2 - (\alpha-\beta')^2}\right)\mathrm{e}^{-\alpha(a-\mathrm{i}\zeta)}. \qquad (11)$$

When releasing the condition $\beta' = 0$ we are in need also of backwards-moving components for the construction of our pulses, since they enter superposition (3') and therefore equation (11). In fact, the spectrum $\Phi = \delta(\beta - \beta')\exp[-a\alpha]$ does obviously entail that $\beta = \beta'$ and

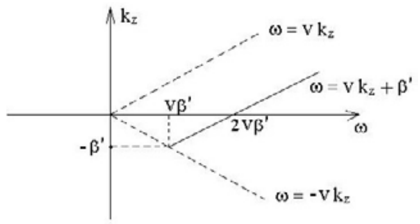


Fig. 4. When releasing the condition $\beta' = 0$ (see the text), which excluded the "backwards-traveling" components, one has to integrate in equation (11) along the half-line $\omega = Vk_z + \beta'$, namely, also along the "non-causal" interval $V\beta' < \omega < 2V\beta'$. We can obtain physical solutions, however, by making negligible the contribution of the unwanted interval, *i.e.*, by choosing small values of a. This can be seen even more easily in the (ω, k_z) plane.

hence, by relations (4), that $\omega = Vk_z + 2V\beta'$. This means (see Fig. 4) that we are now integrating along the continuous line, *i.e.*, also over the interval $V\beta' \le \omega < 2V\beta'$, or $-\beta' \le k_z < 0$, corresponding to the "non-causal" components. Nevertheless, we can obtain physical solutions when making the contribution of that interval negligible, by choosing small values of $a\beta'$: so that the exponential decay of the weight Φ with respect to ω is very slow. Actually, one can go from the (α, β) space back to the (ω, k_z) space by use of equations (4), the weight being re-written (when $\beta' = \beta$) as $\Phi = \exp(-a\omega/V)\exp(-a\beta')$; wherefrom it is clear that[2] for $a \ll 1$ the contribution of the interval $k_z \ge 0$ (or $\omega \ge 2V\beta'$) overruns the $k_z < 0$ contribution. Notice, incidentally, that the corresponding solutions are associated with large frequency bandwidths and therefore to pulses with very short extension in space and in time. Let us mention right now that the spectral weight $\Phi = \exp[-a(\omega - V\beta')/V]$ entails the frequency band-width

$$\Delta\omega = \frac{V}{a},$$

a relation that we shall find to be valid (at least approximately) for all our solutions. We shall discuss this point in Section 5 below.

An analytical expression for integral (11) can be easily found for small positive β' values, when $\beta'^2 \approx 0$. Under such a condition, by using identity (6.616.1) of reference [11] and calling now X the classical [12] X-shaped solution (8)

$$X = X(\rho, \eta) \equiv \left[(a - \mathrm{i}\zeta)^2 + \rho^2(V^2 - 1)\right]^{-1/2}, \tag{12}$$

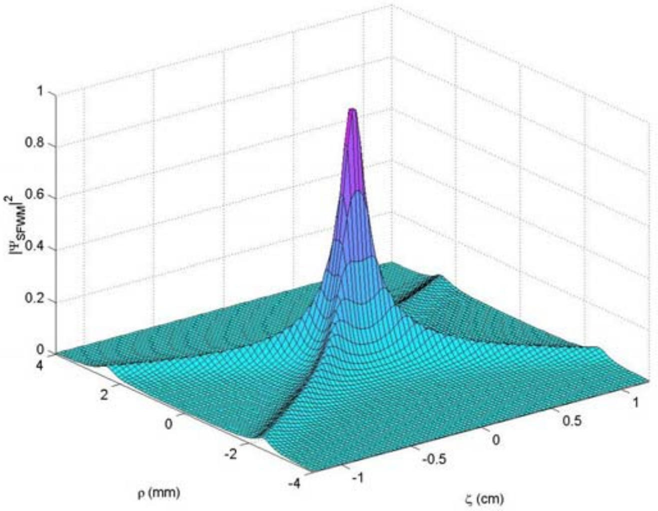


Fig. 5. Representation of our Superluminal focus wave modes (SFWM), equation (13), which are a generalization of the ordinary FWMs. The depicted pulse corresponds to $V = 5c$, $a = 0.001$ m; $\beta' = 1/(100\,\mathrm{m})$, and to arbitrary time t (since these solutions too travel without deforming). Such solutions correspond to high frequency (microwave, optical, ...) pulses: see the text. The meaning of ρ, ζ, etc., is given in the caption of Figure 3.

we obtain the new[3] SLSs [$a > 0$; $\beta' \ge 0$; $V \ge c$]:

$$\Psi_{\mathrm{SFWM}}(\rho, \zeta, \eta) = \mathrm{e}^{-\mathrm{i}\beta'\eta}\, X \times \exp\left[\frac{\beta'(V^2+1)}{V^2-1}\left((a - \mathrm{i}\zeta) - X^{-1}\right)\right] \tag{13}$$

which for $V \to c^+$ reduce to the well-known FWM (focus-wave mode) solutions [15], traveling with speed c:

$$\Psi_{\mathrm{FWM}}(\rho, \zeta, \eta) = \frac{\mathrm{e}^{-\mathrm{i}\beta'\eta}}{a - \mathrm{i}\zeta} \exp\left[-\frac{\beta'\rho^2}{a - \mathrm{i}\zeta}\right]. \tag{14}$$

Our solutions (13) are a generalization of them for $V > c$; we shall call equations (13) the *Superluminal focus wave modes* (SFWM). See Figure 5. Such modes travel without deforming.

Let us emphasize that, when setting $\beta' > 0$, the spectrum (7) results to be constituted (*cf.* Fig. 4) by angular frequencies $\omega \ge V\beta'$. Thus, our new solutions can be used to construct high frequency pulses (*e.g.*, in the microwave or in the optical regions): *cf.* also Section 5.2.

We are going now to build up suitable superpositions of $\Psi_{\mathrm{SFWM}}(\rho, \zeta, \eta)$ in order to get *finite* total energy pulses, in analogy with what is currently attempted [16] for the c-speed FWMs.

3.3 The Superluminal "splash pulses" (SSP)

In the case of the c-speed FWMs, in reference [16] suitable superpositions of them were proposed (the SPs and

[2] One can easily show that the condition $a \ll 1$ should be actually replaced with the condition $a\beta' \ll 1$. In fact (see Fig. 4), the non-causal interval is $\Delta\omega_{\mathrm{NC}} = V\beta'$, while the total spectral band-width is $\Delta\omega = V/a$, so that the non-physical components bring a negligible contribution to the solution in the case of spectrum (7), provided that $\Delta\omega_{\mathrm{NC}}/\Delta\omega \ll 1$, which just means $a\beta' \ll 1$.

[3] Notice that another, slightly different solution — called the FXW — appeared however as equation (4.4) in reference [14].

the "MPS pulses") which possess finite *total* energy (even without truncating them).

Let us analogously go on from our solutions (13) to finite total energy solutions, by integrating our SFWMs (13) over β':

$$\Psi(\rho,\zeta,\eta) \equiv \int_0^{\infty} \mathrm{d}\beta'\, B(\beta')\, \mathrm{e}^{-\mathrm{i}\beta'\eta}\, X \times \exp\left[\frac{\beta'(V^2+1)}{V^2-1}\,\left((a-\mathrm{i}\zeta)-X^{-1}\right)\right], \qquad (15)$$

where it must be still $a \ll 1$, while the weight-functions $B(\beta')$ must be bumped in correspondence with *small* positive values of β' since equation (13) was obtained under the condition $\beta'^2 \approx 0$. In the following, for simplicity, we shall call β, instead of β', the integration variable.

First of all, let us choose in equation (15) the simple weight-function $[\beta' \equiv \beta]$:

$$B(\beta) = \mathrm{e}^{-b\beta} \qquad (16)$$

with $b \gg 0$ for the above-named reasons. Let us recall that such weight (16) is the one yielding in the $V \to c^+$ case the ordinary (c-speed) Splash Pulses [16]; and notice that this choice is equivalent to inserting into equation (3') the spectral weight

$$\Phi(\alpha,\beta;k) \equiv \mathrm{e}^{-a\alpha}\,\mathrm{e}^{-b\beta}. \qquad (7")$$

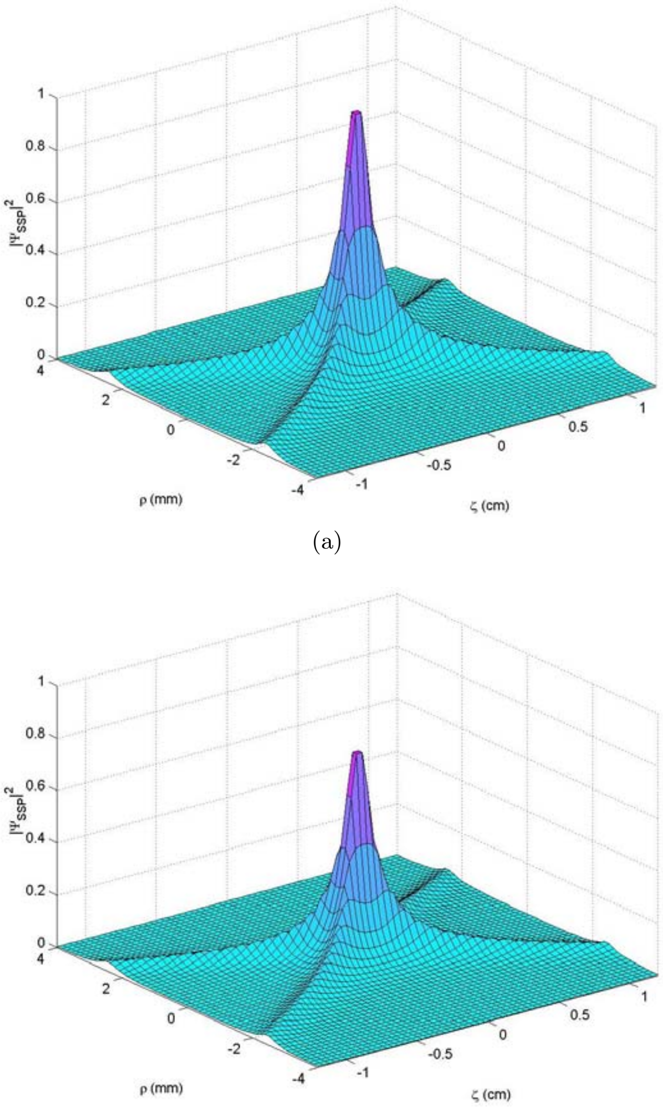
Our *Superluminal Splash Pulses* (SSP) will therefore be:

$$\Psi_{\mathrm{SSP}}(\rho,\zeta,\eta) = X\int_0^{\infty}\mathrm{d}\beta\,\mathrm{e}^{-\beta(b+\mathrm{i}\eta)}\,\mathrm{e}^{\beta Y} = \frac{X}{b+\mathrm{i}\eta-Y}, \qquad (17)$$

with

$$Y \equiv \frac{V^2+1}{V^2-1}\,\left((a-\mathrm{i}\zeta)-X^{-1}\right).$$

Let us repeat that our SSPs have *finite* total energy, as one can easily verify; we shall come back to this result also from a geometric point of view. They however get deformed while traveling, and their amplitude decreases with time: see Figures 6a and 6b. It is worth mentioning that, due to the form (7") of the SSP spectrum, our solution (17) can be regarded as *the finite energy version of the classical X-shaped solution.*



(b)

Fig. 6. Representation of our Superluminal splash pulses (SSP), equation (17). They are suitable superpositions of SFWMs (*cf.* Fig. 5), so that their total energy is *finite* (even without any truncation). They however get deformed while propagating, since their amplitude decreases with time. In (a) we represent, for $t = 0$, the pulse corresponding to $V = 5c$, $a = 0.001$ m, and $b = 200$ m. In (b) it is depicted the same pulse after having traveled 50 meters.

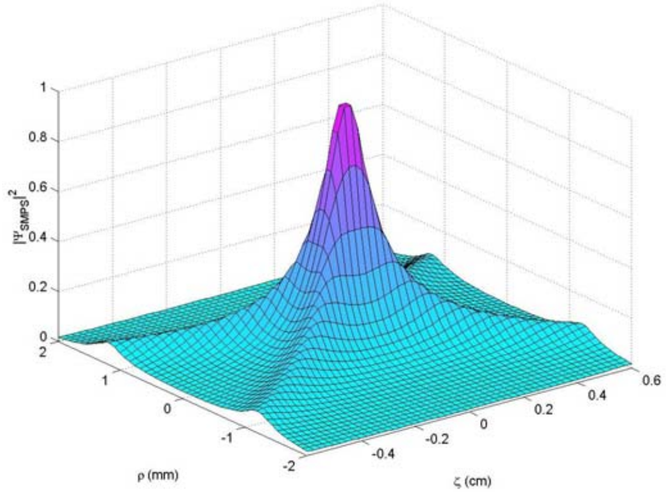
3.4 The Superluminal "modified power spectrum" (SMPS) pulses

In connection with equation (15), let us now go on to a more general choice for the weight-function:

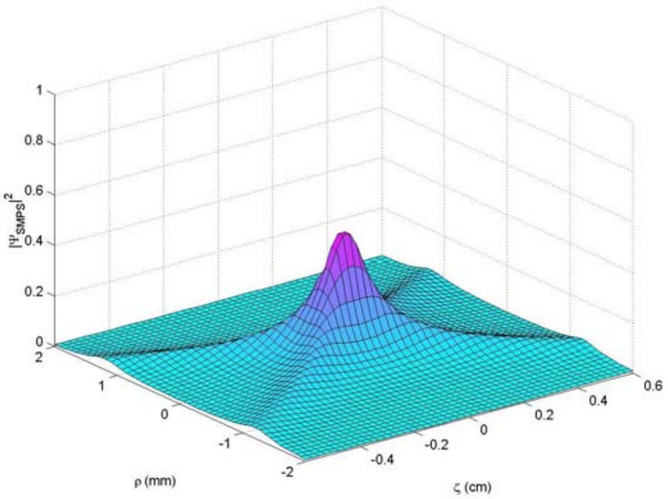
$$\begin{cases} B(\beta) = \mathrm{e}^{-b(\beta-\beta_0)} & \text{for } \beta \geq \beta_0 \\ B(\beta) = 0 & \text{for } 0 \leq \beta < \beta_0 \end{cases} \qquad (16')$$

which for $V \to c^+$ yields the ordinary (c-speed) "modified power spectrum" (MPS) pulses [16]. Such a choice is now equivalent to inserting into equation (3') for $\beta \geq \beta_0$ the spectrum

$$\Phi = \mathrm{e}^{-a\alpha}\,\mathrm{e}^{-b(\beta-\beta_0)} \qquad \text{for } \beta \geq \beta_0. \qquad (7''')$$



(a)



(b)

Fig. 7. Representation of our Superluminal modified power spectrum (SMPS) pulses, equation (18). Also these beams possess *finite* total energy, and therefore get deformed while traveling; figure (a) depicts the shape of the pulse, for $t = 0$, with $V = 5c$, $a = 0.001$ m, $b = 100$ m, and $\beta_0 = 1/(100\,\text{m})$. In (b), it is shown the same pulse after a 50 meters propagation.

We then obtain the *Superluminal modified power spectrum* (SMPS) *pulses* as follows [for $\beta_0 \ll 1$]:

$$\Psi_{\text{SMPS}}(\rho, \zeta, \eta) = e^{b\beta_0}\, X \int_{\beta_0}^{\infty} d\beta\, e^{-(b+i\eta-Y)\beta} = X\, \frac{\exp[(Y - i\eta)\beta_0]}{b - (Y - i\eta)} \qquad (18)$$

in which the integration over β runs *now* from β_0 (no longer from zero) to infinity.

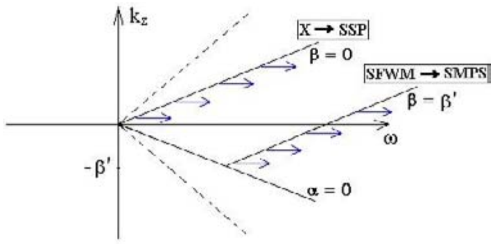


Fig. 8. From a geometric point of view, our infinite total energy SLSs, *i.e.*, the X-solutions, equation (12), and the SFWMs, equation (13), correspond — see the text — to integrations along the $\beta = 0$ axis, or α-axis, and the $\beta = \beta'$ straight-line, respectively. In order to go on to the *finite* total-energy SLSs, we had to replace the $\delta(\beta - \beta')$ factor in the spectrum (7) with the function (16'), which is different from 0 in the region along and below the $\beta = \beta'$ line and suitably *decays* therein. The faster the spectrum decays (below the $\beta = \beta'$ line), the larger the field depth of the pulse results to be. In such a manner we obtained the SMPSs, equation (18), as well as the SSPs, which just correspond to the particular case $\beta' = 0$.

It is worthwhile to emphasize that our solutions (18), like solutions (17), possess a *finite total energy*[4]. Even if this is easily verified, let us address the question from an illuminating geometric point of view. Let us add that their amplitude too (as for the SSPs) decreases with time: see Figures 7a and 7b.

With reference to Figure 8, let us observe that the *infinite* total energy solutions X, in equation (12), and SFWM, in equation (13), correspond to integrations along the $\beta = 0$ axis (*i.e.*, the α-axis) and the $\beta = \beta_0$ straight-line, respectively; that is to say, correspond to a delta factor, $\delta(\beta - \beta_0)$, in the spectrum (7), where $\beta' \equiv \beta_0$.

In order to go on to the *finite* total energy solutions (SMPS), equation (18), we replaced the delta factor with the function (16'), which is zero in the region above the $\beta = \beta_0$ line, while it decays [17] in the region below (as well as along) such a line. The same procedure was followed by us for the solutions SSP, equation (17), which correspond to the particular case $\beta_0 = 0$. The faster the spectrum decay takes place in the region below the $\beta = \beta_0$ line [*i.e.*, $b \gg 1$], the larger the field depth[5] of the corresponding pulse results to be: as we shall see in Section 4.2.3. Let us add that, since $b \gg 1$, even in the present case the non-causal components contribution becomes negligible provided that one chooses $a\beta_0 \ll 1$; in analogy with what we obtained in the previous SFWM case.

It seems important to stress also that, while the X and SSP solutions, equations (12, 18), mainly consist in low-frequency (Bessel) beams, on the contrary our solutions SFWM and SMPS, equations (13, 18), can be constituted by higher frequency beams (corresponding, namely,

[4] One should recall that the first finite energy solution, the MFXW, different from but analogous to our one, appeared as equation (4.6) in reference [14].

[5] The "depth of field" is the distance along which the pulse (approximately) keeps its shape, besides its group-velocity; *cf.* references [2,16].

to $\omega \geq V\beta_0$). This property can be exploited for constructing SLSs in the microwave or optics fields, by suitable choices of the V and β_0 values.

4 Geometric description of the new pulses in the (ω, k_z) plane

4.1 A preliminary analysis of the localized pulses

Let us add some intuitive considerations about the *localized* solutions Ψ to the wave equation, which by our definition [18] must possess the property

$$\Psi(x,y,z;t) = \Psi\left(x, y,\, z + \Delta z_0;\, t + \frac{\Delta z_0}{v}\right) \tag{19}$$

v being the pulse propagation speed, that here can assume *a priori* any [1,2] value: $0 \leq v < \infty$. Such a definition entails that the pulse "oscillates" while propagating, it being required that it resumes (periodically) its shape only after each space interval Δz_0, that is, with a time interval $\Delta t_0 = \Delta z_0/v$ (*cf.* Refs. [18,19]).

Let us write the Fourier-expansion of Ψ

$$\Psi(x,y,z;t) = \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}k_z\, \overline{\Psi}(x,y,k_z;\omega)\, \mathrm{e}^{\mathrm{i}k_z z}\, \mathrm{e}^{-\mathrm{i}\omega t}, \tag{19a}$$

functions $\overline{\Psi}(x,y,k_z;\omega)$ and $\overline{\Psi}(x,y,k_z;\omega)\exp[\mathrm{i}(k_z\,\Delta z_0 - \omega\,\Delta z_0/v)]$ being the Fourier transforms (with respect to the variables z, t) of the l.h.s. and r.h.s. functions in equation (19), respectively; where we used the translation property

$$\mathcal{T}[f(x+a)] = \mathrm{e}^{\mathrm{i}ka}\,\mathcal{T}[f(x)]$$

of the Fourier transformations. From condition (19), we then get [18] the fundamental constraint

$$\omega = vk_z \pm 2n\pi\,\frac{v}{\Delta z_0} \tag{20}$$

linking ω with k_z. Let us explicitly mention that constraint (20) does *not* imply any breakdown of the wave-equation validity. In fact, when inserting expression (19a) into the wave equation, one gets — in cylindrical plane coordinates (ρ, ϕ) — the physical base-solution

$$\Psi(\rho,\phi,k_z;\omega) = J_\mu(k\rho)\,\exp[\mathrm{i}\mu\phi] \tag{19b}$$

with μ an integer and

$$k^2 = \omega^2 - k_z^2 \geq 0. \tag{19c}$$

Therefore, our constraint (20) is compatible with constraint (19c), which followed from the wave equation.

Relation (20) is important, since it clarifies the "spectral origin" of the various localized solutions introduced in the past literature (*e.g.*, for $v = c$), which originated from superpositions performed either by running "along" the straight-lines (20) themselves, or in terms of spectral weights favouring ω, k_z values not far from lines (20). In particular, in our case, in which $v \equiv V > c$, relation (20) brings in a formal, further support of our procedures, as stated in Figures 2, 4 and 8. One may also notice that, when the pulse spectrum does strictly obey equation (20), the pulse depth of field is *infinite* (for instance, the classical X-shaped wave and the SFWM can be regarded as corresponding to equation (20) with $n = 0$ and $n = 1$, respectively[6]). While, when the spectrum is *only* (well) localized in the (ω, k_z) plane, *near* one of the lines (20), the corresponding pulse has a *finite* field depth (as it is the case for our SSP and SMPS solutions). The more "localized" the pulse spectrum is, in the (ω, k_z) plane, in the vicinity of a line (20), the longer the pulse field depth will be. We shall investigate all these points more in detail, in the next subsection.

4.2 Spectral analysis of the new pulses

Let us first recall that throughout this paper it is $\omega \geq 0$, and that, whenever we deal with Superluminal or luminal speeds $V \geq c$, we are confining ourselves (*cf.* Fig. 2b) to the region

$$-\frac{\omega}{V} \leq k_z \leq \frac{\omega}{V}; \qquad [\omega \geq 0]. \tag{21}$$

We are going now to generalize, among the others, what performed in reference [18] for $V = c$.

4.2.1 Generalized X-shaped waves

In the case of the classical X-shaped wave, *the spectrum* $\Phi(\alpha,\beta) = \delta(\beta)\exp[-a\alpha]$ corresponds, because of equations (4), to $\Phi(\omega,k_z) = \delta(\omega - Vk_z)\exp[-a(\omega + Vk_z)/(2V)]$, which imposes the linear constraint

$$\omega = Vk_z; \tag{20a}$$

starts from $\omega = 0$; possesses the (frequency) width

$$\Delta\omega = \frac{V}{a},$$

and results to be bumped for low frequencies.

Notice that this spectrum does exactly lies *along* one of the straight-lines in Figure 4. Actually, equation (20a) agrees with equation (20) for $\Delta z_0 \to \infty$, in accordance with the known fact that the pulse moves rigidly.

In the case of the *generalized* X-pulses, while the straight-line (20a) remains unchanged and the pulse goes on being non-oscillating, the spectrum bump moves towards higher frequencies with increasing m or/and V/a (*cf.* Sect. 3.1).

[6] On a more rigorous ground, the classical X-shaped solution does actually correspond to equation (20) with $\Delta z_0 \to \infty$. For such a reason, it does not oscillate while propagating, and travels rigidly. Analogously, the SSPs will not oscillate: *cf.* Section 4.2.

4.2.2 Superluminal focus wave modes

In the case of the SFWMs, the spectrum $\Phi(\alpha,\beta) = \delta(\beta-\beta')\exp[-a\alpha]$ corresponds (because of Eqs. (4)) to $\Phi(\omega,k_z) = \delta(\omega - Vk_z - 2V\beta'))\exp[-a(\omega+Vk_z)/(2V)]$, which imposes the linear constraint

$$\omega = Vk_z + 2V\beta'. \tag{20b}$$

The minimum value of ω is given (see Fig. 4 and relation (21)) by the intersection of the straight-lines (20b) and $\omega = -Vk_z$. This spectrum starts from $\omega_{\min} = V\beta'$ and possesses the (frequency) width

$$\Delta\omega = \frac{V}{a}\cdot$$

Notice that, once more, the spectrum runs exactly along the line (20b). By comparing equation (20b) with equation (20), one gets that for these oscillating solutions the *periodicity* space and time intervals are

$$\Delta z_0 = \frac{\pi}{\beta'};\qquad \Delta t_0 = \frac{\pi}{V\beta'}\cdot$$

Let us recall from Section 3.2 and Figure 4 that it must be $a\beta' \ll 1$ in order to make negligible the non-causal component contribution (in the two-dimensional expansion). As mentioned in Section 3.2, the relation $\omega \geq V\beta'$ can be exploited for obtaining high frequency SLSs.

4.2.3 Superluminal splash pulses

In the case of the SSPs, the spectrum $\Phi(\alpha,\beta) = \exp[-b\beta]\exp[-a\alpha]$ corresponds (because of Eqs. (4)) to $\Phi(\omega,k_z) = \exp[-b(\omega - Vk_z)/(2V)]\exp[-a(\omega + Vk_z)/(2V)]$. This time the spectrum is *no longer* exactly localized over one of the lines (20); however, if we choose $b \gg 1$ and $a \ll 1$, such a choice together with condition (21) implies $\Phi(\omega,k_z)$ to be well localized in the neighborhood of the line

$$\omega = Vk_z, \tag{20c}$$

besides being almost exclusively composed of causal components. All this can be directly inferred also from the form of $\Phi(\alpha,\beta)$, in connection with Figure 8. The spectrum starts from $\omega_{\min} = 0$, with the frequency width

$$\Delta\omega \simeq \frac{V}{a}\cdot$$

Equation (20) can be compared with equation (20c) only when $b \gg 1$; under such a condition, we obtain that $\Delta z_0 \to \infty$. However, since b can be large but not infinite, the pulse is expected to be endowed in reality with a slowly decaying amplitude, as shown below in Section 5.2.

4.2.4 Superluminal modified power spectrum pulses

In the case of the SMPS pulses, the spectrum is $\Phi(\alpha,\beta) = 0$ for $0 \leq \beta < \beta_0$, and $\Phi(\alpha,\beta) = \exp[b(\beta-\beta_0)]\exp[-a\alpha]$ for $\beta \geq \beta_0$. Under the condition $b \gg 1$ it is $\beta \simeq \beta_0$, that is to say, the spectrum is well localized (as it follows from Eqs. (4)) in the vicinity of the straight-line

$$\omega = Vk_z + 2V\beta_0. \tag{20d}$$

To enforce causality, we choose (as before) also $a\beta_0 \ll 1$. Like in the SFDW pulse case, the spectrum starts from $\omega_{\min} = V\beta_0$, with the frequency width

$$\Delta\omega \simeq \frac{V}{a}\cdot$$

Once more, in the case when $b \gg 1$, one can compare equation (20) with equation (20d), obtaining $\Delta z_0 \simeq \pi/\beta_0$ and $\Delta t_0 \simeq \pi/(V\beta_0)$. Under the condition $b \gg 1$, the pulse is expected to possess a long depth of field, and propagate along it (in an oscillating way) with a maximum amplitude almost constant: we shall look more in detail at this behaviour in Section 5.3.

5 Some exact (Superluminal localized) solutions, and their field depth

To inquiring more in detail into the field depth of our SLSs, we can confine ourselves to the propagation straight-line $\rho = 0$. Then, we can find *exact analytic* solutions holding for any value of β', without having to assume β' to be small, as we had on the contrary to assume for the SFWM, the SST and the SMPS solutions (see Sect. 3, subsections 1, 2, 3). In fact, one is confronted with a simple integration of the type

$$\Psi(\rho=0,\zeta,\eta) = \int_0^\infty \mathrm{d}\alpha \int_0^\infty \mathrm{d}\beta\, \mathrm{e}^{-\mathrm{i}\beta\eta}\,\mathrm{e}^{\mathrm{i}\alpha\zeta}\,\Phi(\alpha,\beta). \tag{3”}$$

Let us first study the infinite total energy solutions: namely, our SFWMs (skipping the generalized X-type solutions).

5.1 The case of the Superluminal focus wave modes

In the case of the SFWMs, solution (11) may be integrated for $\rho = 0$, without imposing the small $\beta_0 \equiv \beta'$ approximation[7]. In fact, by choosing Φ like in equation (7), one obtains

$$\Psi_{\rm SFWM}(\rho=0,\zeta,\eta) = \mathrm{e}^{-\mathrm{i}\beta_0\eta}\int_0^\infty \mathrm{d}\alpha\,\mathrm{e}^{\mathrm{i}\alpha\zeta}\,\mathrm{e}^{-a\alpha}$$
$$= \mathrm{e}^{-\mathrm{i}\beta_0\eta}\,(a-\mathrm{i}\zeta)^{-1} \tag{11a}$$

[7] Also in the case of the SMPS pulses, below, we shall arrive at analytical solutions without any need of imposing the condition that $\beta_0 \equiv \beta'$ be small.

whose square magnitude $|\Psi|^2 = (a^2 + \zeta^2)^{-1}$ reveals that $\Psi_{\rm SFWM}$ is endowed with an infinite depth of field.

Due to the linearity of the wave equation, both the real and the imaginary part of equation (11a), as well as of all our (complex) solutions, are themselves *solutions* of the wave equation. In the following we shall confine ourselves to investigating the behaviour of the real part.

In the case of equation (11a) it is

$$\mathrm{Re}\,[\Psi_{\rm SFWM}(\rho=0,\zeta,\eta)] = \frac{a\,\cos(\beta_0\eta) + \zeta\,\sin(\beta_0\eta)}{a^2+\zeta^2}\,\cdot \tag{11b}$$

The center C of such a pulse (where the pulse reaches its maximum value, M, *oscillating* in space and time) corresponds to $z = Vt$, that is, to $\zeta = 0$ and $\eta = 2z$; its value being

$$M_{\rm SFWM} = \frac{\cos(2\beta_0 z)}{a}\,\cdot \tag{11c}$$

Notice that: (i) at C one meets the maximum value M of the whole three-dimensional pulse: (ii) quantity M is a periodic function of z (and t), with "wavelength" Δz_0 (and oscillation period Δt_0) given by

$$\Delta z_0 = \frac{\pi}{\beta_0}; \qquad \Delta t_0 = \frac{\pi}{V\beta_0}, \tag{11d}$$

respectively: in agreement with what anticipated in Section 4.2.2.

The delta function entering our spectrum (7), entailing that $\beta = \beta_0$, requires that

$$\omega = Vk_z + 2V\beta_0 \tag{22}$$

which is nothing but the straight-line $\beta = \beta_0$ of Figure 8; this fact implying by the way (as we already saw) an infinite field depth, in accordance with the previous considerations in Section 3.4.

By comparing equation (22) with the important "localization constraint" (20), with $n = 1$, we just obtain the value Δz_0 of equation (11d). In other words, the previously got relations (11d) are exactly what needed for the non-dispersiveness of our SFWMs.

Finally, let us examine the longitudinal localization of our oscillating beams. For simplicity, let us analyse the "dispersion" of the beam when its amplitude is maximum; let us therefore skip considering the oscillations and go on to the pulse *magnitude*: one gets for the pulse half-height full-width the value $D = 2\sqrt{3}a$ in the case of the magnitude itself, and

$$D = 2a \tag{23}$$

in the case of the *square* magnitude. Let us adhere to the latter choice in the following, due to a widespread use.

5.2 The finite total energy solutions

Let us now go on to the *finite* total energy solution.

5.2.1 The case of the Superluminal splash pulses

In the case of the SSPs with $\rho = 0$, one has to insert into equation (3") the spectrum (7"), namely $\Phi = \exp[-a\alpha]\exp[-b\beta]$. By integrating, we obtain

$$\Psi_{\rm SSP}(\rho=0,\zeta,\eta) = [(a - \mathrm{i}\zeta)(b+\mathrm{i}\eta)]^{-1}\,, \tag{17a}$$

whose *real part* is

$$\mathrm{Re}\,[\Psi_{\rm SSP}(\rho=0,\zeta,\eta)] = \frac{ab+\eta\zeta}{(ab+\eta\zeta)^2 + (a\eta - b\zeta)^2}\,\cdot \tag{17b}$$

Let us explicitly observe that the chosen spectrum, by virtue of equations (4), entails that these solutions (17a, 17b) do *not* oscillate, which correspond to $\Delta z_0 \to \infty$ and $\Delta t_0 \to \infty$ in equations (20): in agreement with what anticipated in Section 4.2.3. Actually, the SSPs are the *finite energy* version of the classical X-shaped pulses.

The maximum value M of equation (17b) (a not oscillating, but slowly decaying, solution) still corresponds to putting $z = Vt$, that is, to setting $\zeta = 0$ and $\eta = 2z$:

$$M_{\rm SSP} = \frac{b}{a}\frac{1}{b^2+4z^2}\,\cdot \tag{17c}$$

Initially, for $z = 0$, $t = 0$, we have $M = (ab)^{-1}$. If we now *define the field-depth Z as the distance over which the pulse's amplitude is 90% at least of its initial value*, then we obtain the depth of field

$$Z_{\rm SSP} = \frac{b}{6} \tag{24}$$

which shows the dependence of Z on b, namely, the dependence of Z on the spectrum localization in the surroundings of the straight-line $\omega = Vk_z$: *cf.* also Section 3.3.

At last, the longitudinal localization will be approximately given by

$$D \approx 2a; \tag{25}$$

namely, it is still given (for $a \ll 1$ and $b \gg 1$) by equation (23). Notice that, since solution (17a) does not oscillate, the same will be true for its real part, equation (17b), as well as for the square *magnitude* of equation (17a): as it can be straightforwardly verified. Of course, equation (25) holds for $t = 0$. During the pulse propagation, the longitudinal localization D seems to increase, while the amplitude M decreases. Indeed, our preliminary calculations verify that the D-increase rate is approximately equal to the M-decrease rate; so much so we obtain (practically) the same field depth, equation (24), when requesting the longitudinal localization to suffer a limited increase (*e.g.*, by 10% only).

5.2.2 The case of the Superluminal modified power spectrum pulses

In the case of the SMPS pulses with $\rho = 0$, one has to insert into equation (3") the spectrum (7'''), namely

$\Phi = \mathrm{e}^{-a\alpha}\,\mathrm{e}^{-b(\beta-\beta_0)}$, with $\beta \geq \beta_0$. By integration, one gets

$$\Psi_{\rm SMPS}(\rho=0,\zeta,\eta) = \mathrm{e}^{-\mathrm{i}\beta_0\eta}\,\left[(a-\mathrm{i}\zeta)(b+\mathrm{i}\eta)\right]^{-1}, \quad (18a)$$

whose real part is easily evaluated. These pulses do oscillate while traveling. Their field depth, then calculated by having recourse to the pulse square magnitude, happens to be still

$$Z_{\rm SMPS} = \frac{b}{6} \quad (26)$$

like in the SSP case. Even the longitudinal localization of the square amplitude results approximately given, for $t = 0$, by

$$D \approx 2a \quad (27)$$

as in the previous cases.

The field depth (26) depends only on b. However, the *behaviour* of the propagating pulse *changes* with the β_0-value variation, besides with b's. Let us examine the maximum amplitude of the real part of equation (18a), which for $z = Vt$ writes (when $\zeta = 0$ and $\eta = 2z$):

$$M_{\rm SFWM} = \frac{1}{ab}\,\frac{\cos(2\beta_0 z) + 2[z/b]\,\sin(2\beta_0 z)}{1+4[z/b]^2}\,. \quad (18b)$$

Initially, for $z = 0$, $t = 0$, one has $M = (ab)^{-1}$ like in the SSP case.

From equation (18b) one can infer that:

(i) when $z/b \ll 1$, namely, when $z < Z$, equation (18b) becomes

$$M_{\rm SMPS} \simeq \frac{\cos(2\beta_0 z)}{ab}, \qquad [\text{for } z < Z] \quad (28)$$

and the pulse does actually oscillate *harmonically* with wavelength $\Delta z_0 = \pi/\beta_0$ and period $\Delta t_0 = \pi/(V\beta_0)$, all along its field depth: In agreement with what anticipated in Section 4.2.4.

(ii) when $z/b > 1$, namely, when $z > Z$, equation (18b) becomes

$$M_{\rm SMPS} \simeq \frac{\sin(2\beta_0 z)}{ab}\,\frac{1}{2\,[z/b]} \qquad [\text{for } z > Z]. \quad (28')$$

Therefore, *beyond* its depth of field, the pulse go on oscillating with the same Δz_0, but *its maximum amplitude decays* proportionally to z (the decay coefficient being $b/2$).

Last but not least, let us add the observation that results of this kind may find application in the other fields in which an essential role is played by a wave-equation (like acoustics, seismology, geophysics, relativistic quantum physics, gravitational waves).

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Appendix A: Further families of "X-type" Superluminal localized solutions

As announced in Section 3.1, let us mention in this appendix that one can obtain new SLSs by considering *for instance* the following modifications (still with $\beta' = 0$) of the spectrum (7), with a, d arbitrary constants:

$$\Phi(\alpha,\beta;k) = \delta(\beta)\,J_0(2d\sqrt{\alpha})\,\mathrm{e}^{-a\alpha} \quad (A.1a)$$

$$\Phi(\alpha,\beta;k) = \delta(\beta)\,\sinh(\alpha d)\,\mathrm{e}^{-a\alpha} \quad (A.1b)$$

$$\Phi(\alpha,\beta;k) = \delta(\beta)\,\cos(\alpha d)\,\mathrm{e}^{-a\alpha} \quad (A.1c)$$

$$\Phi(\alpha,\beta;k) = \delta(\beta)\,\frac{\sin\alpha d}{\alpha}\,\mathrm{e}^{-a\alpha}. \quad (A.1d)$$

Let us call X, as in equation (8), the classical X-shaped solution

$$X \equiv \left[(a-\mathrm{i}\zeta)^2 + \rho^2(V^2-1)\right]^{\frac{1}{2}}.$$

One can obtain from those spectra the new, different Superluminal localized solutions, respectively:

$$\Psi(\rho,\zeta) = X\,J_0(\rho d^2\,\sqrt{V^2-1}\,X^2) \times \exp\left[-(a-\mathrm{i}\zeta)\,d^2\,X^2\right] \quad (A.2a)$$

got by using identity (6.6444) in reference [11];

$$\Psi(\rho,\zeta) = \frac{2d(a-\mathrm{i}\zeta)\sqrt{2(X^{-2}+d^2)}}{(X^{-2}+d^2) - 4d^2(a-\mathrm{i}\zeta)^2}, \quad (A.2b)$$

for $a > |d|$, by using identity (6.668.1) of reference [11];

$$\Psi(\rho,\zeta) = \left[\frac{X^{-2} - d^2 + \sqrt{(X^{-2}-d^2)^2 + 4d^2(a-\mathrm{i}\zeta)^2}}{2[(X^{-2}-d^2)^2 + 4d^2(a-\mathrm{i}\zeta)^2]}\right]^{\frac{1}{2}} \quad (A.2c)$$

by using identity (6.751.3) of reference [11]; and

$$\Psi(\rho,\zeta) = \sin^{-1} 2d\,\left[\sqrt{X^{-2}+d^2+2\rho d\sqrt{V^2-1}} + \sqrt{X^{-2}+d^2-2\rho d\sqrt{V^2-1}}\right], \quad (A.2d)$$

for $a > 0$ and $d > 0$, by using identity (6.752.1) of reference [11].

Let us recall that, due to the choice $\beta' = 0$ and the consequent presence of a $\delta(\beta)$ factor in the weight, all such solutions are completely physical, in the sense that they don't get *any* contribution from the non-causal components (*i.e.*, from waves moving backwards). In fact, these new solutions are functions of ρ, ζ only (and not of η). In particular, solutions (A.2b, A.2c, A.2d), as well as others easily obtainable, are functions of ρ via quantity X only. This may suggest to go on from the variables (ρ,ζ) to the variables (X,ζ) and write down the wave equation itself in the new variables: Some related results and consequences will be exploited elsewhere.

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